

Some Numerical Results on the Block Spin Transformation for the 2D Ising Model at the Critical Point

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We study the block spin transformation for the 2D Ising model at the critical temperature T_c . We consider the model with the constraint that the total spin in each block is zero. An old argument by Cassandro and Gallavotti strongly supports the Gibbsianness of the transformed measure, provided that such model has a critical temperature T'_c lower than T_c . After describing a possible rigorous approach to the problem, we present numerical evidence that indeed $T'_c < T_c$ and study the Dobrushin–Shlosman uniqueness condition.

KEY WORDS: Ising model; renormalization group; finite-size conditions; critical point.

1. INTRODUCTION

In this paper we discuss the block spin transformation for the two-dimensional Ising model at the critical point, trying to show that it is well defined and gives rise to a Gibbs measure corresponding to a translationally invariant finite norm potential. If one wants to define the renormalization group flow in the space of Hamiltonians this is of course an essential point. Here we study only the first step of this flow. Similar results, based on a computer-assisted proof, have been obtained by Kennedy⁽¹⁴⁾ for the majority rule renormalization group transformation.

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Our aim would be to use a rigorous approach to the subject, but, as will appear clear in the sequel, this is presently a very difficult task. Hence we will mainly analyze the problem from a numerical point of view. However, as we will explain, this will be partially achieved by combining theoretical perfectly rigorous ideas with numerical tools in order to “measure” via a computer some interesting theoretical quantities.

The problem of considering well-defined renormalization group transformations (RGT) and, in particular, the study of RGT in the framework of the modern rigorous approach to statistical mechanics, has attracted the attention of several authors.

Recent and less recent papers have been dedicated to this subject. We want to quote in particular, among the “old” papers, one by Cassandro and Gallavotti⁽²⁾ (which, to our knowledge, is, together with ref. 9, one of the first attempts to explicitly treat the above-mentioned problem), a paper by Girffiths and Pearce,⁽¹⁰⁾ and one by Israel.⁽¹³⁾ For recent results we quote the monumental paper by van Enter *et al.*,⁽⁷⁾ where the problem is discussed in a very clear and complete way. This book-length article includes very interesting discussions and examples. It contains a self-contained exposition of the general setup and a very extensive and up-to-date bibliography. Thus we refer the interested reader to ref. 7 for a review on the subject and for reference to other recent papers.

It is worthwhile to remark that the majority of the examples considered in the above papers concerning the rigorous approach to RGT deal with the region far (and often very far) from the critical point. In some sense, ref. 2 constitutes an exception and some crucial ideas developed in the present paper go back, in fact, to that work.

In the following section we present a general critical discussion about the definition of RGT. In Section 3 we give the definitions concerning the models considered in this paper and the Monte Carlo procedure used to study their equilibrium properties. Section 4 is devoted to the Dobrushin–Shlosman uniqueness condition and to the exact definition of a related numerical quantity that we have studied to analyze the RGT (see next section). Sections 5 and 6 contain the numerical results and, finally, Section 7 is devoted to the conclusions.

2. A CRITICAL DISCUSSION ABOUT THE RGT

The main question arising when rigorously discussing the RGT can be explained by considering, for example, the Ising model at magnetic field h and inverse temperature β . Let

$$v = T_{(b)}^{\mu_{\beta,h}}$$

be a measure arising from the application of a renormalization group transformation $T_{(b)}$, defined “on scale b ,” to the Gibbs measure $\mu_{\beta,h}$ of the Ising model. Transformations of this kind are always trivially defined in finite volume, but, of course, we are interested in taking the thermodynamic limit or, rather, in defining directly the transformation in the infinite-volume situation.

The main “pathology” that can take place, which is the main part of the discussions contained in ref. 7, is that ν can be non-Gibbsian. This means that the conditional probabilities of ν can be incompatible with the Gibbs prescription corresponding to any absolutely summable potential. This non-Gibbsianness is detected via the violation of a necessary condition, namely the property of *quasilocality* for the conditional probabilities of ν .

The Dobrushin–Landford–Ruelle (DLR) theory of Gibbs measures is based on the conditional probabilities π_A for the behavior of the system in a finite box $A \subset\subset \mathbf{Z}^d$, subject to a specific configuration in the complement of A (we use the notation $A \subset\subset \mathbf{Z}^d$ to denote a *finite* subset A of \mathbf{Z}^d). For simplicity let us only consider Ising-like systems. The configuration space of the system in this case is $\Omega = \{-1, 1\}^{\mathbf{Z}^d}$; we use $\Omega_A = \{-1, 1\}^A$ to denote the configuration space in $A \subset \mathbf{Z}^d$. According to ref. 7, a probability measure whose conditional probabilities $\{\pi_A\}_{A \subset\subset \mathbf{Z}^d}$ satisfy

$$\lim_{A' \uparrow \mathbf{Z}^d} \sup_{\omega_1, \omega_2 \in \Omega : (\omega_1)_{A'} = (\omega_2)_{A'}} |\pi_A f(\omega_1) - \pi_A f(\omega_2)| = 0 \quad (1)$$

(namely the conditional expectations in A of any cylindrical function f corresponding to different boundary conditions ω_1, ω_2 , coinciding in $A' \supset A$, tend to coincide as A' tends to \mathbf{Z}^d) is called *quasilocal*. Condition (1) can be seen as a continuity property in the conditioning, infinite-volume, configuration ω .

Kozlov⁽¹⁵⁾ has shown that a quasilocal probability measure on Ω which also satisfies a so-called *nonnullity* condition, i.e., a sort of absence of hard-core exclusion, is *Gibbsian*, in the sense that its conditional probabilities can be obtained, via the Gibbs prescription, from an absolutely summable potential (see ref. 7 for more details).

It is useful to make at this point some remarks about this notion of quasilocality.

1. Kozlov’s theorem (i.e., the fact that nonnullness together with quasilocality imply Gibbsianness) is proved in ref. 15 by using an approach which can be considered somehow artificial. Starting *only* from some nice continuity properties of the conditional probabilities $\{\pi_A\}$, one gets a series, representing the interaction of a point x with the rest of the world,

which is *a priori* only *semiconvergent*. One can insist on extracting the many-body potentials from that series (for instance, via the Moebius inversion formula) pretending that they are absolutely summable. This can be achieved only by regrouping the terms in some suitable order. One can use, for instance, the lexicographic order of the lattice. The resummation will depend in this case on the location of x . The potential will be absolutely summable, but, in general, not translationally invariant. To get translation invariance one needs some stronger properties on how weakly the conditional probabilities depend on far apart configurations.

In some situations to compute the renormalized potentials one can use much stronger methods, based on convergent cluster expansions; in this way a genuine finite-norm, translationally-invariant potential is produced in a very natural way. Each *renormalized coupling constant* is expressed via a convergent series (see, as an example, Cammarota).⁽¹⁾

2. The above notion of quasilocality of a measure μ needs a control *uniform* in the conditioning configuration. To prevent the existence of a Gibbs potential it is sufficient that the condition is violated for *only one* special configuration; in this case even the somehow artificial quantity introduced in ref. 15 cannot be constructed. However, for any reasonable infinite-volume stochastic field μ , a *single* infinite-volume configuration ω is of zero measure; moreover, it can even happen (see below) that the single configuration inducing non-Gibbsianness, as a consequence of nonquasilocality, is very “nontypical” with respect to μ . It is then natural and physically relevant to introduce a weaker notion of quasilocality, for instance, by requiring the validity of a condition like (1) only for μ -almost all ω 's.

A precise definition in this sense has been recently introduced by Fernandez and Pfister,⁽⁸⁾ but this does not prevent the construction of pathological examples. In fact those authors show that, for some interesting examples, a very strong notion of nonquasilocality holds, in the sense that (1) actually fails for μ -almost all configurations ω .

This happens, for instance, for the example of non-Gibbsianness given by Schonmann.⁽²⁴⁾ This example consists simply in considering the *relativization* ν of the measure μ^+ for the Ising model in two dimensions to the line $l = \{x \equiv (x_1, x_2) \in \mathbf{Z}^2 : x_2 = 0\}$ (isomorphic to the one-dimensional lattice \mathbf{Z}^1). Here we are at large inverse temperature β and zero magnetic field; μ^+ is one of the two extremal Gibbs measures, the one obtained via a thermodynamic limit with + boundary conditions; finally, by relativization of μ^+ to l we simply mean the projection on the σ -algebra generated by the spins in l or, simply, the (marginal) distribution of the spins in l deduced by μ^+ by integrating out all the spins in $\mathbf{Z}^2 \setminus l$.

3. There are many cases in which a stochastic field μ shows up the pathology of non-Gibbsianness, as in the Schonmann example; however, at the same time, the measure μ can be, in many respects, very well behaved. For instance, the one-dimensional measure ν of the Schonmann example has exponentially decaying truncated correlations. Moreover, as has been recently shown by Lorinczi and Vande Velde,⁽¹⁶⁾ even this very strong non-Gibbsianness is, in a sense, an *unstable* property.

Let us reanalyze the Schonmann example. Suppose that, instead of considering the one-dimensional sublattice l , one considers a sublattice l_b of sufficiently large spacing b . Namely, starting from μ^+ , one integrates out all the spins outside the set $l_b \equiv \{x \equiv (x_1, x_2) \in \mathbf{Z}^2 : x_2 = 0, x_1 = nb, n \in \mathbf{Z}^1\}$, obtaining the relativized measure ν_b on l_b . One can also see ν_b as obtained via a *decimation procedure* on scale b from ν . In general, given a measure μ on $\Omega = \{-1, 1\}^{\mathbf{Z}^d}$ and an integer $b \geq 2$, the decimation transformation T_b acts on μ so that

$$\nu = T_b \mu$$

is simply the relativization of μ to the sublattice \mathbf{Z}_b^d of \mathbf{Z}^d with spacing b , that is, $\mathbf{Z}_b^d = \{x \in \mathbf{Z}^d : x = by, y \in \mathbf{Z}^d\}$. Lorinczi and Vande Velde show that ν_b is Gibbsian in the strong sense, that the renormalized potential can be computed via a cluster expansion, and that it is absolutely convergent.

Another interesting example given in ref. 7 concerns the decimation transformation applied to the unique Gibbs measure $\mu_{\beta, h}$ for the Ising model at large β and $h \neq 0$, say $h > 0$. They show that, $\forall b$ and for suitable β and h , the renormalized measure $\nu = T_b \mu_{\beta, h}$ arising from a decimation transformation with spacing b is not consistent with any quasilocal specification. In particular it is not the Gibbs measure for any uniformly convergent interaction. As noticed in ref. 7, the nonexistence of the renormalized interaction is a consequence of the presence of a first-order phase transition for the original model in $\mathbf{Z}^d \setminus \mathbf{Z}_b^d$ for particular values of $(\omega_x)_{x \in \mathbf{Z}_b^d}$ and suitable values of h and β . One example is the case where $\omega_x = -1 \forall x$, h uniform and positive, exponentially in β near the value $h^*(b)$, which is needed to compensate, in $\mathbf{Z}^d \setminus \mathbf{Z}_b^d$, the effect of the -1 's in \mathbf{Z}_b^d and to give rise to a degeneracy in the ground state in $\mathbf{Z}^d \setminus \mathbf{Z}_b^d$ (the highly nontrivial part in the proof, given in ref. 7, of the existence of the pathology consists in showing, via the Pirogov–Sinai theory, the persistence of the phase transition at positive temperatures).

On the other hand, from the above analysis it is clear that this pathology comes from the fact that, on a too short spatial scale b (with respect to the thermodynamic parameters and mainly to the magnetic field h), the system is reminiscent of the existence of a phase transition for $h = 0$. It

seems reasonable that this pathology could be eliminated provided one uses an RG transformation defined on a proper scale depending on the thermodynamic parameters. Martinelli and Olivieri⁽²⁰⁾ have shown, exactly for the above example of the Ising model for which in ref. 7 the pathology is found, that, with the same values of β and h , provided one chooses a sufficiently large spacing $b' > b$, the resulting measure $T_{b', \mu_{\beta, h}}$ is Gibbsian in the strong sense and that the renormalized potential, which is absolutely summable, can be computed via a convergent cluster expansion. In particular, taking $b' = b^n$, with n sufficiently large, one shows that, iterating n times the transformation T_b , one goes back to Gibbsian measures; moreover, it has also been shown⁽²⁰⁾ that $T_{b^n, \mu_{\beta, h}}$ converges, as n tends to ∞ , to a trivial fixed point.

Let us now describe an example of pathology discussed in ref. 7 which is particularly relevant in the context of the present paper. It refers to the block averaging transformation (sometimes called Kadanoff transformation). Suppose we partition \mathbf{Z}^2 into square blocks B_i of side 2 (each block containing 4 sites). The block averaging transformation $T_{(2)}^B$ consists, in this case, in the following transformation applied to the Gibbs measure $\mu_{\beta, h}$ for the Ising model at inverse temperature β and magnetic field h ; the new measure is obtained, starting from the original spin variables σ_x , by assigning to any block B_i an integer value m_i and by computing the probability, with respect to the original Gibbs measure $\mu_{\beta, h}$, of the event $\sum_{x \in B_i} \sigma_x = m_i$.

One obtains, starting from $\mu_{\beta, h}(\{\sigma_x\})$

$$v(\{m_i\}) = T_{(2)}^B \mu_{\beta, h}$$

The original system of σ_x variables distributed according to $\mu_{\beta, h}$ is called the *object system*, whereas the new variables m_i distributed according to v constitute the *image system*.

The pathology in the block averaging transformation for the Ising model at large inverse temperature β and arbitrary magnetic field h is a consequence of the existence of a phase transition for the object system for particular values of the image variable m_i . The authors of ref. 7 show that for the configuration with $m_i = 0, \forall i$, the corresponding object system, a constrained Ising model, exhibits a phase transition with long-range order. As a consequence of this fact they are able to show the violation of the quasilocality condition.

Of course, since the local magnetizations m_i in the blocks B_i are fixed and all equal to zero, the value of h is totally irrelevant. On the other hand, if h is very large and, say, positive, the object system without any constraint is almost Bernoulli with a high probability to have an individual spin equal to $+1$ and the configuration with $m_i = 0, \forall i$, is expected to be

very unlikely and, in a sense, irrelevant. Probably the weaker condition of almost sure quasilocality introduced in ref. 8 is satisfied in that situation. Moreover, even though ν is not Gibbsian, it could have nice properties and its non-Gibbsianness could be unstable with respect to small changes. The situation could be similar to the previously mentioned phenomenon discovered by Lorinczi and Vande Velde for the decimation applied to the measure appearing in the Schonmann example.

It is interesting now to discuss in some detail one of the main ideas contained in ref. 2, which is, in a sense, at the basis of the present paper. We will consider the block averaging transformation we have just discussed and apply it to the Ising model Gibbs measure $\mu_{\beta,h}$. The authors of ref. 2 are concerned, in particular, with the most interesting example where $h = 0$ and $\beta = \beta_c$, i.e., the system is at the critical point. They show that, at least formally, it is possible to compute the renormalized potential and to show that it is absolutely summable, provided the constrained Ising model with all the $m_i = 0$, $\forall i$, is above its critical temperature.

To be more precise, let $H^{(r)}(\{m_i\})$ be the renormalized Hamiltonian corresponding to the renormalized measure $\nu(\{m_i\}) = T_{(2)}^B \mu_{\beta,h}$; suppose we extract from $H^{(r)}(\{m_i\})$ all the many-body potentials $\Phi_A(\{m_i\}_{i \in A})$ for any finite set A of blocks B_i . The authors of ref. 2 show that all the Φ_A 's can be expressed as thermal averages of suitable local observables with respect to the Gibbs measure corresponding to an auxiliary intermediate Hamiltonian, which we call $H^{(6)}(S)$. Now, $H^{(6)}(S)$ corresponds to the constrained Ising model with all the m_i 's set equal to zero. The new (intermediate) local variables S_i , defined for any block B_i , take values in the finite space, containing six states, corresponding to the six spin- σ configurations in B_i such that $\sum_{x \in B_i} \sigma_x = 0$.

The starting point of the present paper is to try to show rigorously, in a strong sense, that the auxiliary model with Hamiltonian $H^{(6)}(S)$ does not undergo a phase transition at $\beta = \beta_c$. *A priori* there are no reasons, as will be clear from the discussion in the following sections, for the critical temperature to decrease after the introduction of additional constraints to a spin model. We will show, for example, that, as a consequence of a remark due to Kasteleyn, a particular constrained model obtained from the Ising zero-field model has *exactly* the same critical temperature as the original Ising model!

In particular, to detect the absence of phase transition, we will use the idea of exploiting a *finite-size condition* which goes back to Dobrushin and Shlosman.⁽⁴⁻⁶⁾ The basic point is that if one is able to verify a condition involving mixing properties of (finite-volume) Gibbs measures in a suitable set of finite regions, then one can deduce nice properties (typical of the one-phase region) for the *infinite-volume* system. That can be done, for example,

by using a computer. One can show, for example, uniqueness of the infinite-volume Gibbs measure, analyticity of the infinite-volume thermodynamic and correlation functions, and exponential decay of truncated correlations. In refs. 4–6 the authors avoid the use of cluster expansion; in refs. 5 and 6 they use conditions referring to arbitrary shapes.

In ref. 4 the authors introduce a somehow weak condition implying only uniqueness of the infinite-volume Gibbs state and some decay properties of the infinite-volume truncated correlations. This condition refers to a region $V \subset\subset \mathbf{Z}^d$ and is usually called $\text{DSU}(V)$ from Dobrushin–Shlosman uniqueness condition [see (19), (20) below]. In refs. 5 and 6 they treat the so-called *completely analytical* interactions, proving, on the basis of a stronger condition, much stronger results, in particular uniform analyticity and exponential decay of truncated correlations *for any* finite or infinite volume A with constants uniform in A .

Olivieri and Picco^(22,23) consider similar finite-size conditions, but only for sufficiently regular regions, and get similar results of strong type (such as Dobrushin–Shlosman complete analyticity) by using a block decimation procedure and the theory of the cluster expansion. In a series of papers Martinelli and Olivieri^(17–20) developed a critical analysis of the known finite-size conditions, getting new results both for the equilibrium (Gibbs state) and for the nonequilibrium (Glauber dynamics) situation. The theory developed in refs. 17–19 allows, contrary to the Dobrushin–Shlosman analysis, the treatment, for quite general lattice systems, of almost the whole one-phase region (see, in particular, ref. 18 for more details).

In a recent paper Martinelli *et al.*⁽²¹⁾ showed that, *in two dimensions*, two finite-volume mixing conditions of *a priori* different strength called, respectively, weak mixing (WM) and strong mixing (SM) conditions, are in fact equivalent for sufficiently regular domains A (see ref. 21 for more details). In ref. 4 the authors show that if there exists a region $V \subset\subset \mathbf{Z}^d$ such that their finite-size condition $\text{DSU}(V)$ is satisfied, then weak mixing holds for any finite or infinite $A \subset \mathbf{Z}^d$. Then, combining the results in ref. 4 with the ones in ref. 21, one gets that in two dimensions, if there exists a finite region $V \subset\subset \mathbf{Z}^2$ such that $\text{DSU}(V)$ is satisfied for the constrained Ising system with Hamiltonian $H^{(6)}(S)$, then, for a large class of regular domains, including, for instance, any cube, the strong mixing condition is satisfied for this constrained system.

From strong mixing, using the results obtained in refs. 22 and 23, one can easily make completely rigorous the above-mentioned argument introduced by Cassandro and Gallavotti, and compute the renormalized potentials *as convergent series* via the cluster expansions. In this way, after proving $\text{DSU}(V)$, the Gibbsianness of the renormalized measure would be proven in the strongest possible sense.

3. DEFINITION OF THE MODELS AND OF THE HEAT BATH DYNAMICS

In the following we will give precise definitions about the models that we are going to discuss: the usual 2D Ising model and some "restricted" models obtained from the Ising model by imposing some "extensive" restrictions.

Suppose we partition \mathbf{Z}^2 into 2×2 squared blocks B_i , each containing 4 sites. Each block B_i can be characterized by the coordinates of its lower left-hand site y_i ; namely $B_i \equiv B_{y_i}$, where $y_i = 2x_i$, $x_i \in \mathbf{Z}^2$, and for $x \equiv (x^{(1)}, x^{(2)}) \in \mathbf{Z}^2$,

$$B_{2x} = \{z \in \mathbf{Z}^2 : 2x^{(j)} \leq z^{(j)} < 2(x^{(j)} + 1), j = 1, 2\}$$

The formal Hamiltonian associated to the usual Ising model in zero magnetic field is given by

$$H^{\text{Ising}} \equiv - \sum_{\langle x, y \rangle} \sigma_x \sigma_y \quad (2)$$

where the sum runs over the pairs of nearest-neighbor sites in \mathbf{Z}^2 , and $\sigma_x \in \{-1, +1\}$.

In the following we will consider a system enclosed in a finite squared region \mathcal{A} with various boundary conditions; if not explicitly specified, it will be understood that the boundary conditions are periodic.

In the original Ising model there are, in each block B_i , 16 allowed configurations. Instead of the original σ_i variables to describe such configurations, we can use the block variables, say $S_i \in \{1, \dots, 16\}$. In each block there will be a self-interaction and the mutual interaction between blocks deriving from (2) is again of nearest-neighbor type. The Hamiltonian of the Ising model, expressed in terms of block variables S_i , will be denoted by $H^{(16)}(S) \equiv H^{\text{Ising}}(\sigma)$.

We will consider a modified model in which in each block B_i the sum $m_i = \sum_{x \in B_i} \sigma_x$ is constrained to be zero. Now the block variables S_i will assume only 6 different values, corresponding to the following six block configurations:

$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ + & + \end{bmatrix}, \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} - & + \\ + & - \end{bmatrix} \quad (3)$$

The corresponding Hamiltonian will be denoted by $H^{(6)}(S)$.

It is easy to convince oneself that the model with Hamiltonian $H^{(6)}(S)$ has four periodic ground states.⁽⁷⁾ The first one is given by

$$\begin{array}{cccccccc}
 + & + & + & + & + & + & + & + \\
 - & - & - & - & - & - & - & - \\
 - & - & - & - & - & - & - & - \\
 + & + & + & + & + & + & + & +
 \end{array} \tag{4}$$

The second one is obtained from the first one by interchanging the + with the -; the last two ground states are obtained from the first two by interchanging the rows with the columns. Note that the last two block configurations in (3) are quite different from the first four; in fact they are the only ones that carry a nonzero internal energy and they are absent from the $T=0$ ground-state structure. We call them *turnons*, since, as we will see, they play an important role by allowing the layered ground states to break and mix, destroying long-range order. They are

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}, \quad \begin{bmatrix} - & + \\ + & - \end{bmatrix} \tag{5}$$

By further restricting the allowed block configurations so as to forbid the presence of turnons, we define a new model whose Hamiltonian will be denoted by $H^{(4)}(S)$.

In the following we shall denote the three models introduced before by $\mathcal{S}^{(n)}$, with $n=4, 6, 16$. In Section 3.1 we show that $\mathcal{S}^{(4)}$ with periodic boundary conditions is exactly equivalent to two uncoupled Ising models (in a smaller volume); hence its critical temperature is exactly the same as in $\mathcal{S}^{(16)}$.

We have studied the three models $\mathcal{S}^{(n)}$ by a Monte Carlo procedure, based on a suitable *heat bath dynamics*, whose invariant distribution is the finite-volume Gibbs measure. This dynamics has been used to compute mean values (with respect to the Gibbs measure) of some relevant observables as time averages; the mean value will be denoted by $\langle \cdot \rangle$ in the following.

We have built a discrete-time heat bath dynamics based on locally equilibrating the S_i block variables. We suppose that \mathcal{A} is a cube of even side size, so that it can be exactly partitioned into $N(\mathcal{A})$ B_i blocks, ordered in a lexicographic way, and we define $\Omega_{\mathcal{A}}^{(n)} = \{1, \dots, n\}^{N(\mathcal{A})}$. The dynamics in a finite volume \mathcal{A} is given by a Markov chain defined below.

We perform a complete update of all the $N(\mathcal{A})$ block variables by successively updating each one of them. For any $i=1, \dots, N(\mathcal{A})$, we choose at random the new block configuration S'_i in B_i , given the configuration

$\{S_k\}_{k=1}^{N(A)}$ in A , according to the equilibrium Gibbs measure in B_i with $S|_{B_i^c}$ boundary conditions ($B_i^c = A \setminus B_i$). The related transition probability in the $\mathcal{J}^{(n)}$ model is then given by

$$P^{(n)}(S \rightarrow S') = \frac{\exp[-\beta H^{(n)}(S'_i | S_{B_i^c})]}{\sum_{S''_i=1}^n \exp[-\beta H^{(n)}(S''_i | S_{B_i^c})]}$$

This is the most efficient method as far as local updates of block variables are concerned (since locally we bring at equilibrium the basic block) and is implemented by means of a simple lookup table. It is straightforward to build up the dynamics so that one can move from one model to another simply by changing the number of allowed states [the six blocks of (3) are stored in the first six positions of the tables, with the two turnons in positions 5 and 6].

In order to characterize the critical point of the system we computed two different quantities. First we considered the specific heat C_A as defined from the equilibrium energy fluctuations:

$$C_A \equiv |A|^{-1} \beta^2 (\langle H^2 \rangle - \langle H \rangle^2) \tag{6}$$

We also considered a correlation length ξ defined by measuring zero-momentum correlation functions. We define the sums over planes

$$\tilde{m}(t) \equiv \sum_{s=1}^L \sigma_{(t,s)} \tag{7}$$

where $x = (x^{(1)}, x^{(2)}) \equiv (t, s)$, and the correlations

$$G(t) \equiv \frac{1}{|L-t|} \sum_{t_0=1}^{L-t} \langle \tilde{m}(t_0) \tilde{m}(t_0+t) \rangle \tag{8}$$

A t -dependent correlation length (which we will plot for $t = 5$, where we get a fair estimate for its limit as $t \rightarrow \infty$) can be defined by

$$\xi(t) \equiv \left[\log \frac{G(t)}{G(t+1)} \right]^{-1} \tag{9}$$

For the constrained models, which have a pathological behavior at odd separations, we found it practical to define the correlation length by a distance 2 ratio

$$\xi_{(2)}(t) = \left[\frac{1}{2} \log \frac{G(t)}{G(t+2)} \right]^{-1} \tag{10}$$

Note that in the $\mathcal{J}^{(16)}$ model $\xi(t)$ coincides with $\xi_{(2)}(t)$ for $t \rightarrow \infty$.

Four-State Restriction and Full Ising Model: Proof of the Equivalence

Let us denote by \mathcal{B}_L the family of 2×2 blocks partitioning the cube of (even) side size L and $\{S_\alpha\}_{\alpha \in \mathcal{B}_L}$ the generic configuration of the $\mathcal{J}^{(4)}$ model. We have

$$H^{(4)}(S) = \sum_{\langle \alpha, \beta \rangle} w(S_\alpha, S_\beta) \tag{11}$$

where $\langle \alpha, \beta \rangle$ denotes a pair of nearest-neighbor blocks and $w(s_\alpha, s_\beta)$ is the interaction energy between the two blocks (the self-interaction vanishes).

We want to show that it is possible to associate to each block α an invertible map $\phi_\alpha: S_\alpha \rightarrow (\rho_\alpha, \tau_\alpha) \in \{-1, +1\}^2$, so that, for any pair $\langle \alpha, \beta \rangle$ and any choice of S_α, S_β ,

$$w(S_\alpha, S_\beta) = \rho_\alpha \rho_\beta + \tau_\alpha \tau_\beta \tag{12}$$

We first observe that all four block configurations are of the form

$$\begin{bmatrix} \sigma_1 & \sigma_2 \\ -\sigma_2 & -\sigma_1 \end{bmatrix} \tag{13}$$

Hence there is a simple way to define ϕ_α on the blocks contained in a 4×4 square, so that (12) is verified at least for the blocks in the square. This definition can be visualized in the following picture:

$$\begin{bmatrix} -\tau_2 & -\rho_2 \\ \rho_2 & \tau_2 \end{bmatrix} \begin{bmatrix} -\rho_3 & -\tau_3 \\ \tau_3 & \rho_3 \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} \rho_1 & \tau_1 \\ -\tau_1 & -\rho_1 \end{bmatrix} \begin{bmatrix} \tau_4 & \rho_4 \\ -\rho_4 & -\tau_4 \end{bmatrix}$$

Moreover, if L is a multiple of 4, there is a partition of the lattice into 4×4 squares and it is immediate to check that (12) is satisfied for all pairs $\langle \alpha, \beta \rangle$ if ϕ_α is defined in each 4×4 square as in (14), by periodic extension.

To complete the proof it is sufficient to observe that (12) implies the following identity for the partition functions:

$$Z_{\mathcal{J}^{(4)}, L}(\beta) = Z_{\mathcal{J}^{(16)}, L/2}(\beta)^2 \tag{15}$$

The model $\mathcal{J}^{(4)}$ has four ground states, exactly coinciding with the ones of $\mathcal{J}^{(6)}$ (the turnons in this case are absent at $T=0$). These four states correspond, via the map ϕ_α , to the four ground states of the two independent Ising models (all $+1$ or all -1 for each one of the two Ising systems).

4. THE DOBRUSHIN–SHLOSMAUN UNIQUENESS CONDITION

Let us define the *variation distance* between two probability measures μ_1 and μ_2 on a finite set Y as⁴

$$\text{Var}(\mu_1, \mu_2) = \frac{1}{2} \sum_{y \in Y} |\mu_1(y) - \mu_2(y)| = \sup_{X \subset Y} |\mu_1(X) - \mu_2(X)| \quad (16)$$

Given a metric $\rho(\cdot, \cdot)$ on Y , the *Kantorovich–Rubinstein–Ornstein–Vasserstein distance with respect to ρ* between two probability measures μ_1, μ_2 on Y , which we denote by $V_\rho(\mu_1, \mu_2)$, is defined as

$$V_\rho(\mu_1, \mu_2) = \inf_{\mu \in K(\mu_1, \mu_2)} \sum_{y, y' \in Y} \rho(y, y') \mu(y, y') \quad (17)$$

where $K(\mu_1, \mu_2)$ is the set of joint representations of μ_1 and μ_2 , namely the set of measures on the Cartesian product $Y \times Y$ whose marginals with respect to the factors are, respectively, given by μ_1 and μ_2 . This means that, $\forall B \subset Y$:

$$\begin{aligned} \mu(B \times Y) &= \sum_{y \in B, y' \in Y} \mu(y, y') = \mu_1(B) \\ \mu(Y \times B) &= \sum_{y \in Y, y' \in B} \mu(y, y') = \mu_2(B) \end{aligned}$$

For the particular case

$$\rho(y, y') = \begin{cases} 1 & \text{iff } y \neq y' \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

it is possible to show that $V_\rho(\mu_1, \mu_2)$ coincides with the variation distance $\text{Var}(\mu_1, \mu_2)$.

A result by Dobrushin and Shlosman⁽⁴⁾ concerning the uniqueness of the infinite-volume Gibbs measures generalizes previous results by Dobrushin based on a “one-point condition” on Gibbs conditional distributions (see ref. 3).

Let us consider a spin system on \mathbf{Z}^d with single-spin space \mathcal{S} and finite-range interaction. We generalize in an obvious way the notation introduced in Section 2. Given a metric ρ on \mathcal{S} , we associate to it a metric ρ_A on $\Omega_A \equiv \mathcal{S}^A$, for any $A \subset \subset \mathbf{Z}^d$, by defining

$$\rho_A(S_A, S'_A) = \sum_{x \in A} \rho(S_x, S'_x)$$

⁴ A much more general framework can also be considered.

We say that condition $DSU_\rho(\Lambda, \delta)$ is satisfied if there exist a finite set $\Lambda \subset\subset \mathbb{Z}^d$ and a $\delta > 0$ such that the following is true: for any $y \in \partial^+ \Lambda$ (the set of points outside Λ whose spins interact with the spins inside Λ) there is a positive number α_y such that, for any pair of boundary conditions $\tau, \tau' \in \Omega'_\Lambda$ with $\tau'_x = \tau_x, \forall x \neq y$:

$$V_{\rho_\Lambda}(\mu_\Lambda^\tau, \mu_\Lambda^{\tau'}) \leq \alpha_y \rho(\tau_y, \tau'_y) \tag{19}$$

and

$$\sum_{y \in \partial^+ \Lambda} \alpha_y \leq \delta |\Lambda| \tag{20}$$

where μ_Λ^τ is the Gibbs measure in Λ with boundary conditions τ outside Λ . We say that $DSU(\Lambda, \delta)$ is satisfied if (19) and (20) hold with ρ given by (18).

Theorem 1 (Dobrushin–Shlosman⁽⁴⁾). Let $DSU_\rho(\Lambda, \delta)$ be satisfied for some ρ, Λ , and $\delta < 1$; then $\exists C > 0, \gamma > 0$ such that condition $WM_\rho(\Lambda, C, \gamma)$ holds for every Λ .

By $WM_\rho(\Lambda, C, \gamma)$ we mean a particular mixing property of μ_Λ^τ saying that the influence at $x \in \Lambda$ of a change in the conditioning spins τ decays as

$$C e^{-\gamma \text{dist}(x, \partial \Lambda)}$$

See ref. 18 for a precise definition.

Theorem 1 implies, in particular, the uniqueness of infinite-volume Gibbs measures. Then (19), (20) provide an example of a finite-size condition: one supposes that some properties of a *finite-volume* Gibbs measure are true and then deduces properties of *infinite-volume* distributions.

We observe now that

$$V_{\rho_\Lambda}(\mu_\Lambda^\tau, \mu_\Lambda^{\tau'}) \geq \sum_{x \in \Lambda} V_\rho(\mu_x^\tau, m_x^{\tau'})$$

where μ_x^τ is the probability distribution of the spin s_x with respect to the measure μ_Λ^τ . Hence, if ρ is given by (18), then

$$V_{\rho_\Lambda}(\mu_\Lambda^\tau, \mu_\Lambda^{\tau'}) \geq \sum_{x \in \Lambda} \text{Var}(\mu_x^\tau, \mu_x^{\tau'})$$

It follows that, for given A , $DSU(A, \delta)$ is certainly *not* satisfied for any $\delta < 1$, provided that

$$\sup_{\tau, \tau': \tau_x = \tau'_x, \forall x \neq y} \sum_{x \in A} \text{Var}(\mu_x^\tau, \mu_x^{\tau'}) \geq \frac{|A|}{|\partial^+ A|}, \quad \forall y \in \partial^+ A$$

This observation will be used in our numerical calculations on model $\mathcal{J}^{(6)}$ in the following way. We try to find a “good” lower bound for the quantity

$$\sup_{y \in \partial^+ A_L} \sup_{\tau, \tau': \tau_x = \tau'_x, \forall x \neq y} \sum_{x \in A_L} \text{Var}(\mu_x^\tau, \mu_x^{\tau'}) \tag{21}$$

by calculating numerically $\sum_{x \in A_L} \text{Var}(\mu_x^\tau, \mu_x^{\tau'})$ for a “large” number of choices of y , τ , τ' and by considering the maximum among these numbers, which we call Δ_L . Here A_L denotes a cube of side size L in \mathbb{Z}^2 and each point of A_L represents a 2×2 block; hence A_L corresponds to a cube of side size $2L$ in the original lattice.

We consider a “reasonable” indication that $DSU(A, \delta)$ is satisfied for A large enough if $\Delta_L |\partial^+ A_L|/|A_L|$ is decreasing in L and there exists and L_c such that

$$\mathcal{M}_L \equiv \Delta_{L/2} \frac{|\partial^+ A_{L/2}|}{|A_{L/2}|} = \Delta_{L/2} \frac{8}{L} < 1, \quad L > L_c \tag{22}$$

Of course, checking inequality (22) does not imply the validity of the Dobrushin–Shlosman uniqueness condition. Nevertheless the two quantities \mathcal{M}_L and

$$\frac{1}{|A|} \sum_{y \in \partial^+ A_L} \sup_{\tau, \tau': \tau_x = \tau'_x, \forall x \neq y} \rho_A(\mu_A^\tau, \mu_A^{\tau'})$$

are, in our opinion, strictly related; hence we think that the validity of inequality (22) is a good test for the absence of long-range order.

We have analyzed, in particular, the equivalent of “Dobrushin boundary conditions” for our model; namely conditions such that on the upper half of the boundary there is a configuration tending to create in the bulk one of the coexisting phases, whereas on the lower half there is a configuration favoring another, different, phase. We did not find any particular instability with respect to a local change of the conditioning spins; in other words, also considering these boundary conditions does not change the estimate Δ_L of the quantity (21).

5. NUMERICAL RESULTS: THE PHASE TRANSITION

We present here a first set of numerical results which give numerical evidence that the $\mathcal{J}^{(6)}$ model undergoes a phase transition at a temperature

$$T_c^{(\mathcal{J}^{(6)})} < T_c^{(\mathcal{J}^{(16)})}$$

The correlation length of the constrained model at the critical point of the full Ising model is a finite, reasonably small number, which we estimate.

This numerical evidence is not meant to constitute a large-scale simulation. We do not try here to estimate critical exponents or to determine with high precision the position of critical points (for large-scale simulations of the 2D Ising model see, for example, ref. 12 and references therein). The main point here is to show in a nonambiguous manner that the two critical temperatures are different and that the correlation length of the constrained model at the critical point of the Ising model is finite.

Our results have been obtained for cubes containing 400^2 lattice points (i.e., the region A_L with $L = 200$, according to the notation of Section 4). In the case of model $\mathcal{J}^{(6)}$ we have performed 2×10^5 full sweeps (that is, full update of all lattice sites) of our heat bath block algorithm per each value of the inverse temperature β . In the case of $\mathcal{J}^{(4)}$ and $\mathcal{J}^{(16)}$ models we have used 10^5 sweeps per each value of β . We have simulated smaller volumes to check that everything was well compatible with the expected finite-size behavior (but we will not report in detail these data). All the runs discussed in this section use periodic boundary conditions.

Let us start by discussing our simulations for the full 2D Ising model. As we said, very large scale simulations exist⁽¹²⁾ and the results we are presenting here are just meant to set the frame for showing numerically the different behavior of the two relevant models. So we have simulated the full 2D Ising model in the same conditions that we have used to study the constrained $\mathcal{J}^{(6)}$ model.

In Fig. 1 we show the specific heat of the model. The point closer to criticality is the one at $\beta = 0.4400$ (as is fortunately obvious from the picture). Here, as well as in the following, the point size is of the order of magnitude of the statistical error, except for the point closer to criticality. The mild (logarithmic) divergence of the specific heat is at an inverse critical temperature which we can estimate from our data to be at $\beta_c = 0.440 \pm 0.001$, in agreement with the known exact value.

In Fig. 2 we show the correlation length $\xi \equiv \xi(5)$ [see (9)], which diverges at T_c with a critical exponent $\nu = 1$. We plot ξ for $\xi \ll L$.

The second set of numerical simulations refer to model $\mathcal{J}^{(6)}$. We show respectively the specific heat and the correlation length in Figs. 3 and 4.

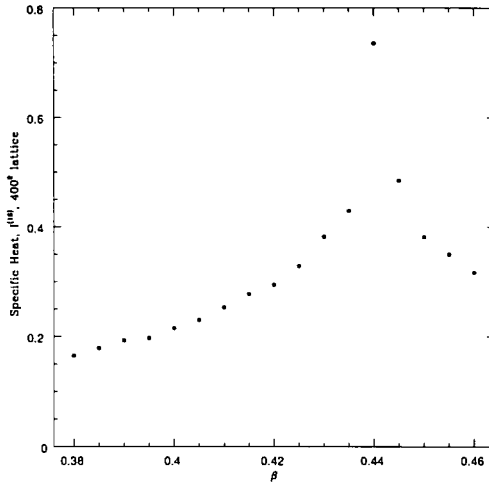


Fig. 1. The specific heat (as computed from energy fluctuations) as a function of β for the Ising model.

These two figures strongly suggest that $T_c^{(\mathcal{J}^{(6)})} < T_c^{(\mathcal{J}^{(16)})}$: the critical temperature of the constrained model is smaller than the one of the original model. At the critical temperature of the Ising model the constrained model does not show a critical behavior. We estimate

$$\beta_c^{(\mathcal{J}^{(6)})} = 0.4775 \pm 0.0025 \tag{23}$$

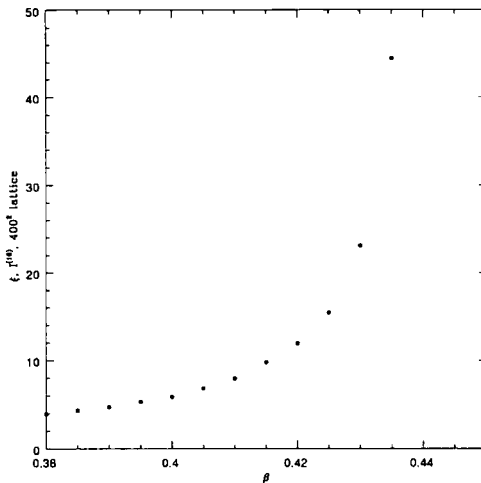


Fig. 2. The correlation length ξ as a function of β for the Ising model.

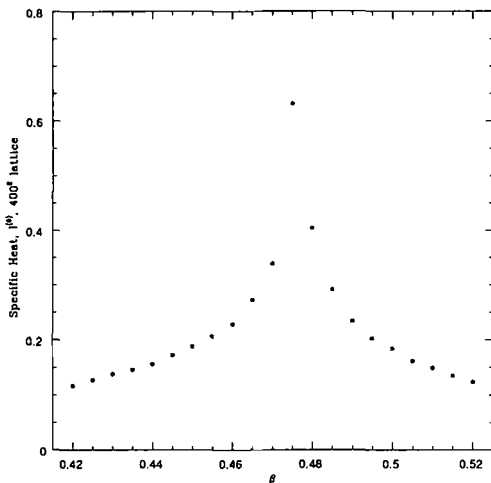


Fig. 3. As in Fig. 1, but for the six-block state constrained Ising model, $\mathcal{I}^{(6)}$.

The correlation length of model $\mathcal{I}^{(6)}$ is finite at the critical temperature of the Ising model; we estimate [see (10) for the definition and note that it is defined in units of the original lattice]:

$$\xi_{(2)} \equiv \xi_{(2)}(5) \simeq 11.5 \pm 0.5 \quad \text{if } \beta = \beta_c^{(\mathcal{I}^{(6)})} \tag{24}$$

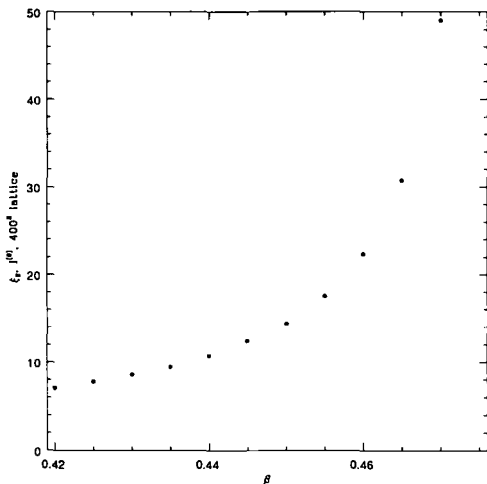


Fig. 4. As in Fig. 2, but for the six-block state constrained Ising model, $\mathcal{I}^{(6)}$.

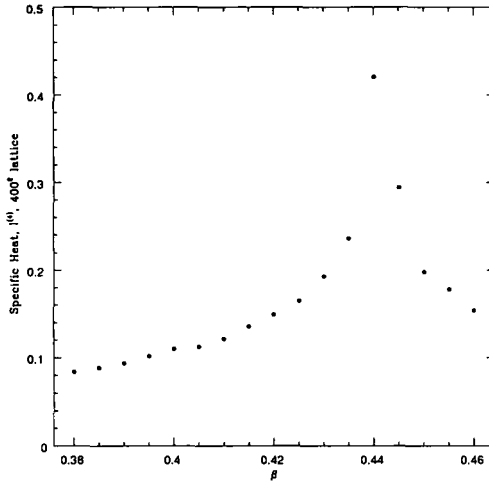


Fig. 5. As in Fig. 1, but for the four-block state constrained Ising model, $\mathcal{I}^{(4)}$.

This is in agreement with the numerical calculations related to the DSU condition that we shall discuss in the following section. In fact we find that inequality (22) of Section 4 is satisfied for $L > L_c$, with $L_c \simeq 5\xi$.

As we have already said, we have also simulated the model restricted to four states (by forbidding turnons), which is equivalent to two decoupled Ising models. In Fig. 5 and 6 we show the specific heat and the

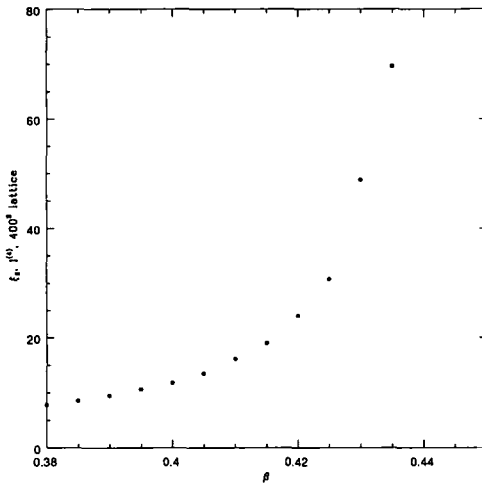


Fig. 6. As in Fig. 2, but for the four-block state constrained Ising model, $\mathcal{I}^{(4)}$.

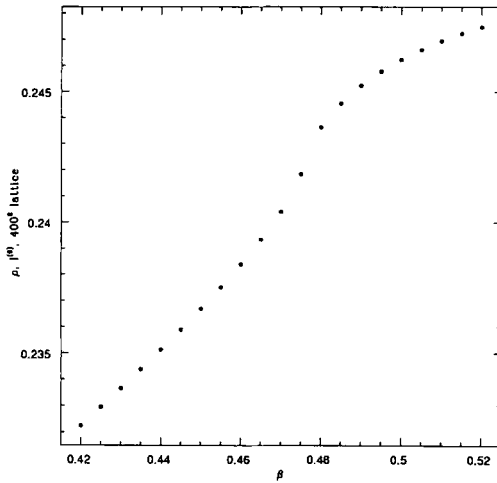


Fig. 7. Plot of ρ as a function of β for the six-block state model, $\mathcal{J}^{(6)}$.

correlation length; they clearly show criticality at $T_c^{(\mathcal{J}^{(4)})} = T_c^{(\mathcal{J}^{(16)})}$. Note that the correlation length is larger than in model $\mathcal{J}^{(16)}$, as one expects, because the effective lattice spacing of the two independent Ising models is half the spacing in the original lattice. The ratio between the two correlations lengths should indeed be exactly 2 if we could calculate the limit of $\zeta_{(2)}(t)$ as $t \rightarrow \infty$.

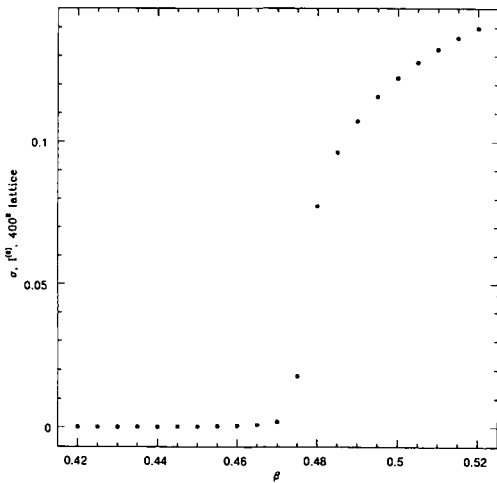


Fig. 8. Plot of σ as a function of β for the six-block state model, $\mathcal{J}^{(6)}$.

In order to better characterize the nature of the transition for the constrained model $\mathcal{J}^{(6)}$ we will present some more data. As discussed in Section 3, at $T=0$ the $\mathcal{J}^{(6)}$ model has four ground states, with broken translational symmetry. We define ρ_i , $i=1, \dots, 4$, as the projection of a given configuration over the i th ground state (we count the number of blocks which fit the ground-state pattern and normalize with the total number of blocks). We define

$$\rho \equiv \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4) \quad (25)$$

Let us recall again that the ground states do not include turnons. The turnon density is $(1-4\rho)$, and tends to zero for $\beta \rightarrow \infty$. In Fig. 7 we plot ρ as a function of T for model $\mathcal{J}^{(6)}$. The density of turnons at $\beta = 0.4407$, the critical point of the full 2D Ising model, is close to 0.06, and determines the finite correlation length.

Spontaneous symmetry breaking is signaled by the nonvanishing, in the infinite-volume limit, of σ defined by

$$\sigma \equiv \frac{1}{4} \sum_{i=1}^4 (\rho_i - \rho)^2 \quad (26)$$

In the symmetric phase, $\sigma \rightarrow 0$ in the infinite-volume limit. In the broken phase the system aligns in one of the four ground states (tunneling, at finite volume, among the different layered ground states), and σ is nonzero. We plot σ in Fig. 8. The location of the critical point is signaled with high precision from the drastic change in σ .

6. NUMERICAL RESULTS: THE DOBRUSHIN-SHLOSMAN CONDITION

As discussed at the end of Section 4, we have used the inequality (22) to obtain some reasonable insight about the validity of the Dobrushin-Shlosman uniqueness condition, which is too difficult to verify also numerically for volumes as large as needed in our case. However, even the task of checking numerically the inequality (22) is indeed a difficult one with respect to the calculations of the previous section. In fact in this case one has to control the whole distribution. This is quite difficult compared with the simple task of computing averages of some observable quantities. The internal energy, for example, is peaked close to the expectation value; hence, in order to compute its expectation value, one just needs to explore a very restricted part of the phase space. A computation of \mathcal{M}_L [see (22)] demands, on the contrary, as we will see, a very large statistics.

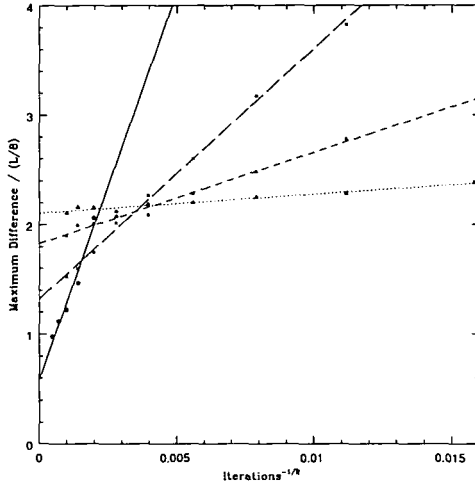


Fig. 9. Plot of \mathcal{M}_L as a function of the inverse square root of the number of sweeps, for different lattice sizes. Straight lines are best linear fits. Empty triangles for $L=8$, filled triangles for $L=16$, crosses for $L=32$, and filled dots for $L=64$.

In our simulations we considered a sequence of cubes of side size L and, after selecting L and the number I of Monte Carlo sweeps (we define a Monte Carlo sweep as the update of all lattice sites), we chose N different random boundary conditions by assigning to the boundary sites the value $+$ or $-$ with equal probability. We shall call τ_k , $k=1, \dots, N$, these boundary conditions.

For each τ_k we selected (randomly) a boundary pair of adjacent sites belonging to the same boundary block and considered the four boundary conditions τ'_k , $j=1, \dots, 4$, differing from each other only in the chosen boundary block, by changing in all possible ways the values of the two spins. We did four Monte Carlo runs with the four boundary conditions and, during these runs, recorded the number of times each block variable visited each of the six allowed states, by constructing

$$\tilde{N}_{\tau'_k}(i, S_i) \tag{27}$$

where j ranges over the four different values, i ranges over the $(L/2) \times (L/2)$ blocks, and S_i ranges over the six allowed block configurations. \tilde{N} is normalized in such a way that

$$\sum_{S_i} \tilde{N}_{\tau'_k}(i, S_i) = 1 \tag{28}$$

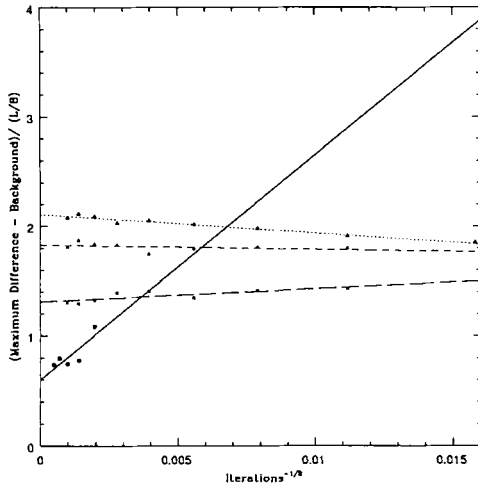


Fig. 10. As in Fig. 9, but for $\tilde{\mathcal{M}}_L$.

The quantity $\Delta_{L/2}$ of (22) was then calculated as

$$\Delta_{L/2} \equiv \max_{k=1, \dots, N} \max_{j \neq l; j, l=1, \dots, 4} \left\{ \frac{1}{2} \sum_{i, S_i} |\tilde{N}_{\tau_k^j}(i, S_i) - \tilde{N}_{\tau_k^l}(i, S_i)| \right\} \quad (29)$$

In our simulations we used $L=8, 16, 32, 64$, N of order 100 and values of I ranging from 4000 to 4×10^6 (for the largest value of L).

In Fig. 9 we plot the quantity \mathcal{M}_L of (22) as a function of the inverse square root of the run length for different lattice sizes and number of iterations. The straight lines are our best linear fit, which turns out to be the right ansatz for the observed behavior.

The inequality (22) is satisfied only on the 64^2 lattice and is violated on smaller lattices. Our best fits give

$$\begin{aligned} \mathcal{M}_8 &= 2.10 + \frac{17}{\sqrt{I}} \\ \mathcal{M}_{16} &= 1.83 + \frac{83}{\sqrt{I}} \\ \mathcal{M}_8 &= 1.32 + \frac{229}{\sqrt{I}} \\ \mathcal{M}_{64} &= 0.58 + \frac{706}{\sqrt{I}} \end{aligned} \quad (30)$$

There is indeed a big contribution due to the fact that we are adding a finite number of positive random numbers. Only in the limit of a large number of iterations, $I \rightarrow \infty$, does this contribution go to zero. In order to minimize this effect we have also run simulations with the same boundary condition τ_k for each k . The average value of the difference

$$\left\{ \frac{1}{2} \sum_{i, S_i} |\tilde{N}_{\tau_k}(i, S_i) - \tilde{N}'_{\tau_k}(i, S_i)| \right\}$$

obtained in these conditions has been subtracted from \mathcal{M}_L in order to define $\tilde{\mathcal{M}}_L$. The contribution we have subtracted from \mathcal{M}_L goes to zero in the limit $I \rightarrow \infty$, making $\tilde{\mathcal{M}}_L$ a good estimator for the inequality (22). We plot $\tilde{\mathcal{M}}_L$ in Fig. 10. Our best fits for $\tilde{\mathcal{M}}_L$ give

$$\begin{aligned} \tilde{\mathcal{M}}_8 &= 2.11 - \frac{17}{\sqrt{I}} \\ \tilde{\mathcal{M}}_{16} &= 1.83 - \frac{4.0}{\sqrt{I}} \\ \tilde{\mathcal{M}}_{32} &= 1.31 + \frac{12.0}{\sqrt{I}} \\ \tilde{\mathcal{M}}_{64} &= 0.60 + \frac{205}{\sqrt{I}} \end{aligned} \tag{31}$$

The constant term coincides, as it should, with the one we get for \mathcal{M} , with very good precision. On the contrary, slopes are smaller, indicating that $\tilde{\mathcal{M}}$ is, as expected, a better estimator than \mathcal{M} when using a finite number of iterations.

7. CONCLUSIONS

From the numerical results on specific heat and correlation length, which are based on “traditional” numerical methods to detect a (second-order) phase transition, we can be reasonably sure that, indeed, the critical temperature of the $\mathcal{J}^{(6)}$ model is strictly less than T_c^{Ising} . The remark on the isomorphism between $\mathcal{J}^{(4)}$ and the full Ising model $\mathcal{J}^{(16)}$ shows that one cannot *a priori* expect any monotonicity property of critical temperatures in terms of imposed constraints. Our numerical results indicate that, by imposing the constraint $m_i = 0, \forall B_i$, one *decreases* the critical temperature, whereas by enhancing again the constraint via the further elimination of the turnons, one *increases* the critical temperature since it goes back to T_c^{Ising} .

To analyze theoretically this apparently strange behavior it seems useful to use some generalized form of Fortuin–Kasteleyn representation of the Ising model sufficiently “elastic” to include, in the same setup of random-cluster models, the three models $\mathcal{J}^{(16)}$, $\mathcal{J}^{(6)}$, $\mathcal{J}^{(4)}$ that we have considered.⁽¹¹⁾

On the other hand, to implement rigorously the Cassandro–Gallavotti program one needs a *strong* notion of absence of phase transition, namely to verify some strong mixing condition SM.

We say that a Gibbs measure μ_A^τ in \mathcal{A} with τ boundary conditions outside \mathcal{A} satisfies a *strong mixing condition* $SM(\mathcal{A}, C, \gamma)$ if the influence at $x \in \mathcal{A}$ of a *local* change in $y \in \mathcal{A}^c$ of the value of the conditioning spin configuration τ decays as

$$Ce^{-\gamma|x-y|}$$

Of course $SM(\mathcal{A}, C, \gamma)$ implies $WM(\mathcal{A}, C, \gamma)$ (see Section 4); moreover, $SM(\mathcal{A}, C, \gamma)$ is interesting when it is valid for a class of volumes \mathcal{A} invading \mathbf{Z}^d with C and γ independent of \mathcal{A} .

As we said in Section 2, in ref. 2 it is proven that, in two dimensions, WM implies SM at least for sufficiently regular regions \mathcal{A} . SM would be certainly largely sufficient to compute, via the Cassandro–Gallavotti method, the renormalized potentials, whereas, in general, WM alone will not.

In Section 6 we analyzed a sort of lower bound (involving variation distance) for the quantity appearing in the Dobrushin–Shlosman uniqueness condition (implying WM and so, since we are in 2D, SM). This is only an (almost) necessary condition to be verified in order to satisfy the true condition DSU. It only gives an indication in the sense of the possibility to verify DSU (which involves the Vasserstein distance).

It is clear that one needs to consider squared regions containing about 30×30 blocks. This rules out, at least with present computers, the possibility of a computer-assisted proof. Then it would be important to be able to find a “Monte Carlo” method to “measure” the quantity (Vasserstein distance) appearing in DSU.

It would also be interesting, in general, to find an algorithm, easily implementable in a computer, to evaluate the Vasserstein distance between two Gibbs measures with different boundary conditions. This will be the object of further investigations.

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